Blacktown Boys' High School

2023 Year 12

Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided for this paper
- In Questions in Section II, show all relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 3 – 6)
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7 – 13)
- Attempt Questions 10 – 14
- Allow about 1 hour 45 minutes for this section

Assessor: K.Villanueva

NAME: ______________________________

Students are advised that this is a trial examination only and cannot in any way guarantee the content or format of the 2023 Higher School Certificate Examination.

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Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 – 10 .

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1 What is the derivative of $\tan^{-1}\frac{x}{4}$?

A. $\frac{4}{16+x^2}$

B. $\frac{16}{16+x^2}$

C. $\frac{1}{16+x^2}$

D. $\frac{1}{2(16+x^2)}$

2 If $\sin x = \frac{3}{5}$, where $90° \leq x \leq 180°$, what is the value of $\tan 2x$?

A. $-\frac{7}{24}$

B. $\frac{7}{24}$

C. $-\frac{24}{7}$

D. $\frac{24}{7}$

3 Let $P(x)$ be a polynomial of degree 7. When $P(x)$ is divided by the polynomial $Q(x)$, the remainder is $3x^2 + 7$.

Which of the following is true about the degree of Q?

A. The degree must be 3.

B. The degree could be 3.

C. The degree must be 2.

D. The degree could be 2.

4 If $f(x) = 3 - 6x$, what is the value of $f^{-1}(2)$?

A. -9

B. $-\frac{1}{6}$

C. $\frac{1}{6}$

D. 9

5 The random variable X is such that $X \sim Bin(n, p)$. The mean value of X is 225 and variance of X is 144. What is the value of p?

A. 0.8

B. 0.64

C. 0.6

D. 0.36

6 Which of the following integrals is equivalent to $\int (\sin^2 4x + 1)\, dx$?

A. $-\int \frac{1+\cos 8x}{2}\, dx$

B. $-\int \frac{1+\sin 8x}{2}\, dx$

C. $\int \frac{3-\cos 8x}{2}\, dx$

D. $\int \frac{3-\sin 8x}{2}\, dx$

7 What is the acute angle between the vectors $3\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{i} + 5\boldsymbol{j}$, correct to the nearest degree?

A. 19°

B. 26°

C. 33°

D. 48°

8 A curve has parametric equations $y = \frac{3}{2t}$ and $x = 4t + \frac{1}{t}$.

What is the Cartesian equation for this curve?

A. $2xy + 3y^2 = 18$

B. $2xy - 3y^2 = 18$

C. $3xy + 2y^2 = 18$

D. $3xy - 2y^2 = 18$

9 A stationary box contains 5 felt-tip pens, 6 pencils, 9 ball-point pens, 9 whiteboard markers and 14 permanent markers. What is the minimum number of items that must be chosen randomly from the box to guarantee obtaining 7 items of the same type?

A. 44

B. 32

C. 30

D. 29

10 Given that the roots of $2x^2 - \sqrt{3}x + 1 = 0$ are $\tan\alpha$ and $\tan\beta$, what is the value of $\alpha + \beta$?

A. $\frac{\pi}{6}$

B. $\frac{\pi}{4}$

C. $\frac{\pi}{3}$

D. $\frac{2\pi}{3}$

End of Section I

Section II Answer Booklet
60 marks
Attempt Questions 11–14

Start each question in a SEPARATE booklet. Extra writing booklets are available.

For questions 11–14, your responses should include all relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) Let $P(x) = x^3 - 10x^2 + 13x + 24$.

(i) Show that $P(8) = 0$. **1**

(ii) Hence factor the polynomial $P(x)$ as $A(x)B(x)$, where $B(x)$ is a quadratic polynomial. **2**

(b) The vectors $\boldsymbol{u} = \begin{pmatrix} a \\ -2 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} a-4 \\ a+8 \end{pmatrix}$ are perpendicular. **2**

What are the possible values of a?

(c) (i) Express $\cos x + \sqrt{3}\sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$. **3**

(ii) Hence solve $\cos x + \sqrt{3}\sin x = 1$, for $0 \le x \le 2\pi$. **1**

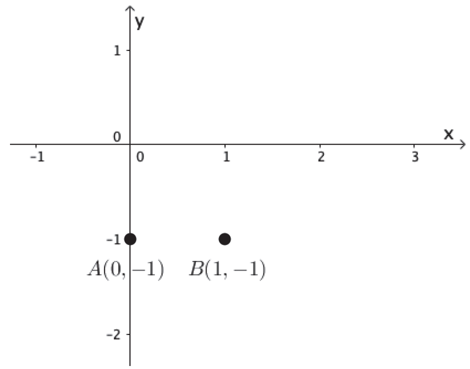
(d) Solve $\dfrac{x}{x-4} < 3$. **3**

(e) Use the substitution $u = \sqrt{x+1}$ to find $\displaystyle\int_0^{15} \frac{x}{\sqrt{x+1}}\,dx$. **3**

Question 12 (16 marks)

(a) A direction field is to be drawn for the differential equation $\dfrac{dy}{dx} = \dfrac{2x - y}{y^2 + x^2}$. **2**

Copy the diagram below into your writing booklet, then clearly draw the correct slopes of the direction field at the points A and B.



(b) Sous vide cooking is a process where food is placed in a vacuum sealed plastic bag or glass jar and immersed in a temperature regulated water bath. A bag of corn, with temperature 25°C, is placed in a water bath that remains at a constant temperature of 85°C. After t minutes, the temperature of the bag of corn, in degrees Celcius, is T.
The temperature of the water can be modelled using the differential equation

$$\frac{dT}{dt} = k(T - 85) \qquad \text{(Do NOT prove this.)}$$

where k is the growth constant.

(i) After 15 minutes, the temperature of the bag of corn is 50°C. By solving the differential equation through integration, find the value of t when the temperature of the bag of corn reaches 70°C. Give your answer to the nearest minute. **3**

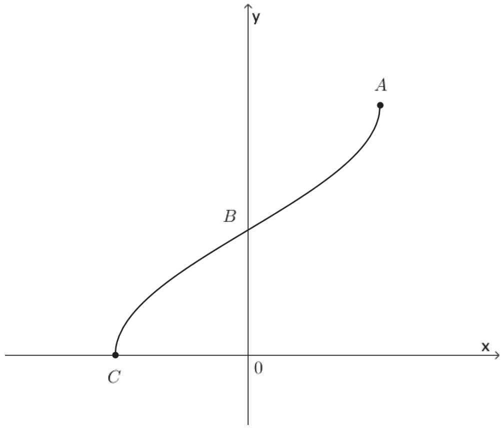
(ii) Sketch the graph of T as a function of t. **1**

(c) A drawer contains 11 different batteries of which 7 are good and 4 are defective. A teacher selects two batteries for the remote control of the projector and then selects another two for the remote control of the air conditioner. If both batteries must be good for each remote control to work, find the probability that:

(i) Both remote controls work. **1**

(ii) Only the remote control for the projector works. **2**

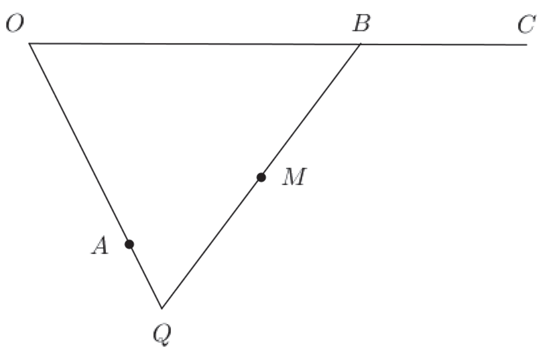
(d) The diagram below shows the graph of $y = \pi + 2\sin^{-1} 3x$.



(i) Find the coordinates of A and C. **2**

(ii) Find the equation of the tangent at B. **2**

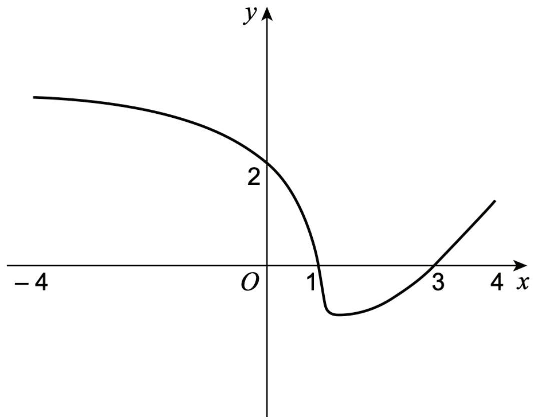
(e) In the diagram below $\overrightarrow{OA} = 3\boldsymbol{a}$, $\overrightarrow{AQ} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$, $\overrightarrow{BC} = \frac{1}{2}\boldsymbol{b}$, and M is the midpoint of QB. **3**



Prove that AMC is a straight line.

Question 13 (14 marks)

(a) The diagram below shows the graph of $y = f(x)$ for $[-4, 4]$. The graph has x–intercepts at $x = 1$, and $x = 3$, and y–intercept at $y = 2$.



(i) Sketch the graph of $y = f(|x|)$. **2**

(ii) Sketch the graph of $y = \dfrac{1}{f(x)}$. **2**

(b) *You may use the information on page 15 to answer this question.*

A toy car factory knows that 3% of the toy cars it produces are faulty. Toy cars are supplied in boxes of 50 cars, and boxes are supplied in pallets of 80 boxes.

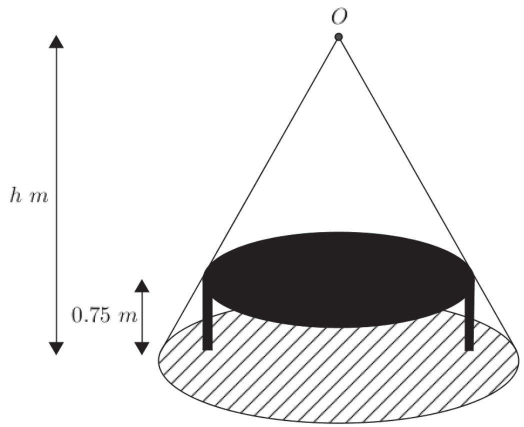
Find the probability, to 4 decimal places, that:

(i) A box of toy cars contains exactly 5 faulty cars. **1**

(ii) A box of toy cars contains at least 1 faulty car. **1**

(iii) A pallet contains between 65 and 70 (inclusive) boxes with at least 1 faulty car. **3**

(c) A small lamp O is placed h metres above the ground. Vertically below the lamp is the centre of a round table of radius $1.5\ m$ and height of $0.75\ m$. The lamp casts a shadow of the table on the ground. Let $S\ m^2$ be the area of the shadow.



(i) Show that $S = \dfrac{2.25\pi h^2}{(h-0.75)^2}$. **2**

(ii) If the lamp is lowered vertically at a constant rate of $\dfrac{8}{27}\ m/s$, find the rate of change of S with respect to time when $h = 3.75\ m$. **3**

Question 14 (15 marks)

(a) (i) Show that $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$. **1**

(ii) Hence show that $\cot 15° = 2+\sqrt{3}$. **1**

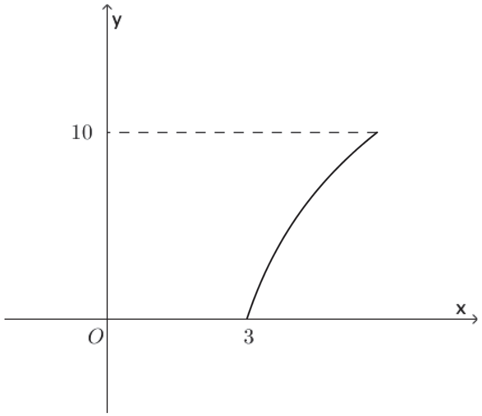
(b) For all integers $n \geq 1$, use mathematical induction to prove that **3**

$$\frac{1}{3\times4\times5}+\frac{2}{4\times5\times6}+\frac{3}{5\times6\times7}+\cdots+\frac{n}{(n+2)(n+3)(n+4)} = \frac{1}{6}-\frac{1}{n+3}+\frac{2}{(n+3)(n+4)}$$

(c) In the centre of Australia, the population of kangaroo was estimated to be 85 000 in 1990. The population growth is given by the logistic equation $\dfrac{dP}{dt} = \dfrac{1}{15}P\left(\dfrac{C-P}{C}\right)$ where t is the number of years after 1990 and C is the carrying capacity. In the year 2020, the population of kangaroo was estimated to be 425 000. **4**

Use the fact that $\dfrac{C}{P(C-P)} = \dfrac{1}{P}+\dfrac{1}{C-P}$ to show that the carrying capacity is approximately 1 137 000.

(d) A bowl is formed by rotating the curve $y = 10\log_e(x-2)$ about the y-axis for $0 \le y \le 10$. **3**



NOT TO SCALE

Find the exact volume of the bowl.

(e) At a book manufacturer, the proportion of pages printed incorrectly by a printer is $p = 0.007$, which is considered to be acceptable. To confirm that the printer is working to this standard, a sample of size n is taken and the sample proportion $\hat{p}$ is calculated. **3**

It is assumed that $\hat{p}$ is approximately normally distributed with $\mu = p$ and $\sigma^2 = \dfrac{p(1-p)}{n}$.

Pages printed by this printer will be shut down if $\hat{p} \ge 0.008$.

The sample size is to be chosen so that the chance of shutting down the printer unnecessarily is less than 0.15%.

Find the approximate sample size required, giving your answer to the nearest thousand.

End of paper

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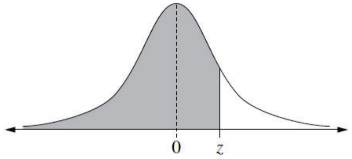
Table of values $P(Z \leq z)$ for the normal distribution $N(0, 1)$



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995

Student Name: ______________________________________

Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word ‘correct’ and drawing an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

Start Here →

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

2023 Yr12 Ext 1 Trial Examination Solutions

SECTION I

1) A

$\tan^{-1}\frac{x}{4}$

$$\frac{d}{dx} = \frac{\frac{1}{4}}{1+\left(\frac{x}{4}\right)^2}$$

$$= \frac{1}{4\left(1+\frac{x^2}{16}\right)}$$

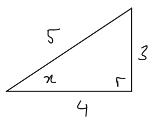
$$= \frac{1}{4+\frac{x^2}{4}}$$

$$= \frac{1}{\frac{16}{4}+\frac{x^2}{4}}$$

$$= \frac{4}{16+x^2}$$

2) C

$\sin x = \frac{3}{5}$



$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$= \frac{2\times\left(-\frac{3}{4}\right)}{1-\frac{9}{16}}$$

$$= \frac{-3/2}{7/16}$$

$$= -\frac{24}{7}$$

3) B

$Q(x)$ can have degree 3, 4, 5, 6

4) C

For $x=2$ for $f^{-1}(x)$, it is $y=2$ for $f(x)$.

$3-6x=2$

$-6x=-1$

$x=\frac{1}{6}$

5) D

$\mu = np = 225$ ①

$Var(x) = np(1-p) = 144$ ②

Sub ① into ②

$225(1-p) = 144$

$1-p = 0.64$

$-p = -0.36$

$p = 0.36$

6) C

$$\int \sin^2 4x + 1\,dx = \int\left(\tfrac{1}{2}(1-\cos 2(4)x)+1\right)dx$$

$$= \int\left(\frac{1-\cos 8x}{2}+1\right)dx$$

$$= \int\left(\frac{1-\cos 8x}{2}+\frac{2}{2}\right)dx$$

$$= \int \frac{3-\cos 8x}{2}$$

7) B

$$\cos\theta = \frac{3\times1 + 4\times5}{\sqrt{3^2+4^2}\times\sqrt{1^2+5^2}}$$

$$= \frac{3+20}{\sqrt{25}\times\sqrt{26}}$$

$$= \frac{23}{5\sqrt{26}}$$

$$= 25.5599\ldots$$

$$\approx 26^\circ$$

8) D

$$y = \frac{3}{2t} \qquad x = 4t + \frac{1}{t}$$

$$2t = \frac{3}{y}$$

$$t = \frac{3}{2y}$$

$$x = 4\left(\frac{3}{2y}\right) + \frac{1}{3/2y}$$

$$x = \frac{6}{y} + \frac{2y}{3}$$

$$x = \frac{18+2y^2}{3y}$$

$$3xy = 18 + 2y^2$$

$$3xy - 2y^2 = 18$$

9) C

Worst case scenario

5 felt tip + 6 pencils + 6 ball-point + 6 whiteboard

+ 6 permanent + 1 final item = 30

10) C

$$2x^2 - \sqrt{3}x + 1 = 0$$

$$\tan\alpha + \tan\beta = -\frac{b}{a} = \frac{\sqrt{3}}{2}$$

$$\tan\alpha\tan\beta = \frac{c}{a} = \frac{1}{2}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$\tan(\alpha+\beta) = \sqrt{3}$$

$$\alpha+\beta = \frac{\pi}{3}$$

QUESTION 11

a) i) $P(x) = x^3 - 10x^2 + 13x + 24$

$P(8) = 8^3 - 10(8)^2 + 13(8) + 24$

$= 512 - 640 + 104 + 24$

$= 0$

① correct solution

ii) $P(x) = (x-8)(ax^2 + bx + c)$

$x^3 - 10x^2 + 13x + 24 = (x-8)(ax^2+bx+c)$

$x^3 = ax^3$ $\qquad 13x = cx - 8bx$ $\qquad 24 = -8c$

$a = 1$ $\qquad 13 = c - 8b$ $\qquad c = -3$

$13 = -3 - 8b$

$16 = -8b$

$b = -2$

$\therefore P(x) = (x-8)(x^2 - 2x - 3)$

② correct solution

① Finds 2 values of a, b or c

OR

significant progress in dividing

b) $\begin{pmatrix} a \\ -2 \end{pmatrix} \cdot \begin{pmatrix} a-4 \\ a+8 \end{pmatrix}$

$a(a-4) - 2(a+8) = 0$

$a^2 - 4a - 2a - 16 = 0$

$a^2 - 6a - 16 = 0$

$(a-8)(a+2) = 0$

$a = 8, -2$

② correct solution

① obtains $a^2 - 6a - 16 = 0$

c) i) $\cos x + \sqrt{3}\sin x = R\cos(x - \alpha)$

$R\cos(x + (-\alpha)) = R\cos x\cos(-\alpha) - R\sin x\sin(-\alpha)$

$\cos x + \sqrt{3}\sin x = R\cos x\cos\alpha + R\sin x\sin\alpha$

$R\cos\alpha = 1$ ①

$R\sin\alpha = \sqrt{3}$ ②

③ correct solution

② Significant progress

① Correct α or R value

$①^2 + ②^2$

$R^2 = 1^2 + (\sqrt{3})^2$

$R^2 = 4$

$R = 2$

$② \div ①$

$\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$

$\therefore \cos x + \sqrt{3}\sin x = 2\cos\left(x - \frac{\pi}{3}\right)$

ii) $\cos x + \sqrt{3}\sin x = 1$

$2\cos\left(x - \frac{\pi}{3}\right) = 1$

$\cos\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$

$x - \frac{\pi}{3} = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$

$x = -\frac{\pi}{3} + \frac{\pi}{3}, \frac{\pi}{3} + \frac{\pi}{3}, \frac{5\pi}{3} + \frac{\pi}{3}$

$= 0, \frac{2\pi}{3}, 2\pi$

① correct solution

d) $\frac{x}{x-4} < 3$

$x(x-4) < 3(x-4)^2$

$x(x-4) - 3(x-4)^2 < 0$

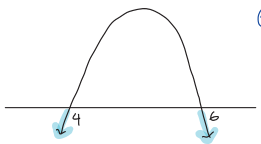
$(x-4)(x - 3(x-4)) < 0$

$(x-4)(x - 3x + 12) < 0$

$(x-4)(-2x+12) < 0$

4 $\qquad$ 6

$\therefore x < 4, x > 6$



③ correct solution

② Makes significant progress and identifies 4 and 6 are 2 key values.

① Multiplies both sides by the square of the denominator.

e) $\int_0^{15} \frac{x}{\sqrt{x+1}}\, dx$

$u^2 = x+1$, $2u\,du = dx$

$x=0$, $u=1$

$x=15$, $u=4$

$\int_1^4 \left(\frac{u^2-1}{u}\, 2u\right) du$

$= 2\int_1^4 (u^2-1)\, du$

$= 2\left[\frac{1}{3}u^3 - u\right]_1^4$

$= 2\left[\left(\frac{64}{3}-4\right) - \left(\frac{1}{3}-1\right)\right]$

$= 2\,[21-3]$

$= 36$

③ correct solution

② correct primitive function

① correct substitution

QUESTION 12

a) $\frac{dy}{dx} = \frac{2x-y}{y^2+x^2}$

$(0,-1)$

$\frac{dy}{dx} = \frac{2(0)-(-1)}{(-1)^2-0^2}$

$= \frac{1}{1}$

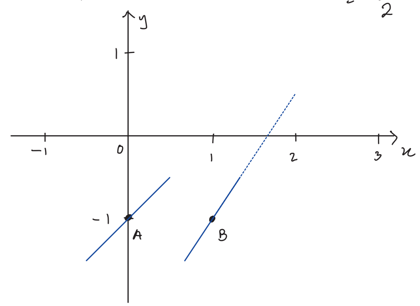
$= 1$

$(1,-1)$

$\frac{dy}{dx} = \frac{2(1)-(-1)}{(-1)^2+(1)^2}$

$= \frac{2+1}{2}$

$= \frac{3}{2}$



② correct solution with graph showing reference point

① correct substitution to calculate

b) i) $\frac{dT}{dt} = k(T-85)$

$\frac{dt}{dT} = \frac{1}{k(T-85)}$

$t = \frac{1}{k}\int \frac{1}{T-85}\, dT$

$kt = \ln|T-85| + C$

when $t=0$, $T = 25°C$

$0 = \ln|25-85| + C$

$C = -\ln 60$

OR

let $C = \ln|T-85|$

$Ae^{kt} = T-85$

$T = Ae^{kt} + 85$

③ correct solution

② Finds the value of k

① Obtains the value of A OR $T = 25 + Ae^{kt}$

$$kt = \ln|T-85| - \ln 60$$
$$= \ln\left|\frac{T-85}{60}\right|$$

when $t = 15$, $T = 50°C$

$$15k = \ln\left|\frac{50-85}{60}\right|$$
$$15k = \ln\left(\frac{7}{12}\right)$$
$$k = \frac{1}{15}\ln\left(\frac{7}{12}\right)$$

$t = ?$ when $T = 70°C$

$$\frac{1}{15}\ln\left(\frac{7}{12}\right)t = \ln\left|\frac{70-85}{60}\right|$$
$$t = \frac{15\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{7}{12}\right)}$$
$$= 38.57987\ldots$$
$$= 39 \text{ minutes}$$

when $t = 0$, $T = 25°C$

$$25 = Ae^{0} + 85$$
$$A = -60$$

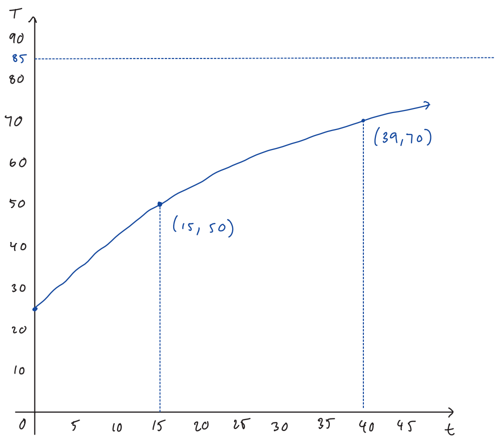
when $t = 15$, $T = 50°C$

$$50 = -60e^{15k} + 85$$
$$-35 = -60e^{15k}$$
$$\frac{35}{60} = e^{15k}$$
$$k = \frac{\ln\left(\frac{7}{12}\right)}{15}$$

$t = ?$ when $T = 70°C$

$$70 = -60e^{kt} + 85$$
$$\frac{-15}{-60} = e^{kt}$$
$$\frac{1}{4} = e^{kt}$$
$$\ln\left(\frac{1}{4}\right) = kt$$
$$t = \ln\left(\frac{1}{4}\right) \Big/ \frac{\ln\left(\frac{7}{12}\right)}{15}$$
$$t = 38.57987\ldots$$
$$t = 39 \text{ minutes}$$

ii)



① correct solution

c) i) Total number of possible selections $= {}^{11}C_2 \times {}^{9}C_2$

$= 1980$

① correct solution

Both remotes work $= {}^{7}C_2 \times {}^{5}C_2$

$= 210$

$$P(\text{both remotes work}) = \frac{210}{1980} = \frac{7}{66}$$

ii) 2 possible outcomes where only projector remote works

1) Air conditioner remote doesn't work with 2 defective batteries

2) Air conditioner remote doesn't work with 1 defective battery and 1 good battery

Total number of outcomes:

$${}^{7}C_2 \times {}^{4}C_2 + {}^{7}C_2 \times {}^{5}C_1 \times {}^{4}C_1 = 546$$

$$P(\text{only projector remote works}) = \frac{546}{1980} = \frac{91}{330}$$

② correct solution

① Determines the number of outcomes for at least 1 scenario where projector won't work

d) i) $A\left(\frac{1}{3}, 2\pi\right)$

$C\left(-\frac{1}{3}, 0\right)$

② correct solution

① 1 correct endpoint

ii) $y = \pi + 2\sin^{-1} 3x$

$$\frac{dy}{dx} = 2 \times \frac{3}{\sqrt{1-9x^2}}$$

when $x = 0$, $\frac{dy}{dx} = \frac{6}{\sqrt{1-0}}$

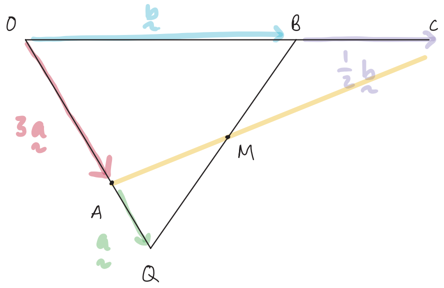
$= 6$

$$y - \pi = 6(x - 0)$$
$$6x - y + \pi = 0$$

② correct solution

① correct differentiation

e)



(3) correct solution

(2) Determines an expression for $\vec{AM}$ and $\vec{MC}$

(1) Determines an expression for $\vec{QB}$

$\vec{OQ} = 4\underset{\sim}{a}$

$\vec{QB} = \vec{QO} + \vec{OB}$

$= -4\underset{\sim}{a} + \underset{\sim}{b}$

$\vec{QM} = \vec{MB} = \frac{1}{2}(-4\underset{\sim}{a} + \underset{\sim}{b})$

$= -2\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$

$\vec{AM} = \vec{AQ} + \vec{QM}$

$= \underset{\sim}{a} + (-2\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b})$

$= -\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$

$\vec{MC} = \vec{MB} + \vec{BC}$

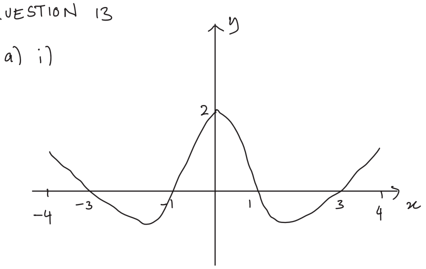
$= -2\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{b}$

$= -2\underset{\sim}{a} + \underset{\sim}{b}$

$\therefore$ AMC is a straight line as $2\vec{AM} = \vec{MC}$ and M is a common point

QUESTION 13

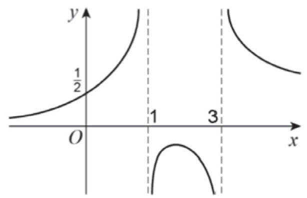
a) i)



(2) correct solution

(1) Recognises all key features OR correct shape

ii)



(2) correct solution

(1) Recognise asymptotes + one part of graph correct

b) i) $P(\text{faulty}) = 0.03$, $P(\overline{\text{faulty}}) = 0.97$

$P(F=5) = {}^{50}C_5 (0.03)^5 (0.97)^{45}$

$= 0.013074...$

$= 0.0131$ (4 d.p)

(1) correct solution

ii) $P(F \geq 1) = 1 - P(F=0)$

$= 1 - {}^{50}C_0 (0.03)^0 (0.97)^{50}$

$= 0.781934...$

$= 0.7819$ (4 d.p)

(1) correct solution

iii) Let X = number of boxes where $F \geq 1$

Let $\hat{p}$ = proportion of boxes where $F \geq 1$

$E(\hat{p}) = p = 1 - (0.97)^{50}$

$Var(\hat{p}) = \dfrac{(1-(0.97)^{50})(1-(1-(0.97)^{50}))}{80}$

$= 0.0021314...$

(3) correct solution

(2) Finds z-score of $X=65$ or $X=70$

(1) Finds $E(\hat{p})$ and $Var(\hat{p})$

$$\sigma(\hat{p}) = \sqrt{\frac{(1-(0.97)^{50})(1-(1-(0.97)^{50}))}{80}}$$

$0.0461672\ldots$

If $X = 65$ then $\hat{p} = \frac{65}{80} = 0.8125$

If $X = 70$ then $\hat{p} = \frac{70}{80} = 0.875$

$$z\text{-score } (x = 65) = \frac{0.8125 - E(\hat{p})}{\sigma(\hat{p})}$$

$$= 0.662058\ldots$$

$$z\text{-score } (x = 70) = \frac{0.875 - E(\hat{p})}{\sigma(\hat{p})}$$

$$= 2.0158329\ldots$$

$$P(65 \leq X \leq 70) = P(0.66 \leq z \leq 2.02)$$

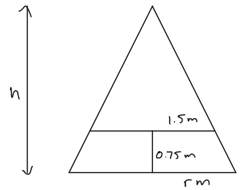
$$= 0.9783 - 0.7454$$

$$= 0.2329$$

C) i) Show that $S = \dfrac{2.25\pi h^2}{(h-0.75)^2}$

② correct solution

① Determines $r = \dfrac{1.5h}{h-0.75}$



$$S = \pi r^2$$

$$\frac{r}{h} = \frac{1.5}{h-0.75}$$

$$r = \frac{1.5h}{h-0.75}$$

$$S = \pi\left(\frac{1.5h}{h-0.75}\right)^2$$

$$S = \frac{2.25\pi h^2}{(h-0.75)^2}$$

ii) $\dfrac{dS}{dt} = \dfrac{dS}{dh} \times \dfrac{dh}{dt}$ $\qquad \dfrac{dh}{dt} = -\dfrac{8}{27}$

$S = \dfrac{2.25\pi h^2}{(h-0.75)^2}$ $\qquad u = 2.25\pi h^2 \quad v = (h-0.75)^2$

$\qquad u' = 4.5\pi h \quad v' = 2(h-0.75)$

$$\frac{dS}{dh} = \frac{(4.5\pi h) \times (h-0.75)^2 - (2.25\pi h^2) \times (2(h-0.75))}{(h-0.75)^4}$$

$$= \frac{(4.5\pi h)(h-0.75)^2 - (4.5\pi h^2)(h-0.75)}{(h-0.75)^4}$$

$$= \frac{(4.5\pi h)(h-0.75)((h-0.75)-h)}{(h-0.75)^4}$$

$$= \frac{4.5\pi h(-0.75)}{(h-0.75)^3}$$

③ correct solution

② obtains $\dfrac{dS}{dt} = \dfrac{\pi h}{(h-0.75)^3}$

① correct differentiation of S

$$= -\frac{3.375\pi h}{(h-0.75)^3}$$

$$\frac{ds}{dt} = -\frac{3.375\pi h}{(h-0.75)^3} \times -\frac{8}{27}$$

$$= \frac{\pi h}{(h-0.75)^3}$$

at $h = 3.75$

$$\frac{ds}{dt} = \frac{3.75\pi}{(3.75-0.75)^3}$$

$$= \frac{5\pi}{36}\ m^2/s$$

QUESTION 14

a) i) $\frac{\sin 2x}{1+\cos 2x} = \tan x$

① correct solution

$$LHS = \frac{2\sin x\cos x}{\cancel{1} + (2\cos^2 x - \cancel{1})}$$

$$= \frac{\cancel{2}\sin x\,\cancel{\cos x}}{\cancel{2}\cos^{\cancel{2}} x}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

ii) $\cot 15° = 2 + \sqrt{3}$

① correct solution

$$\tan 15° = \frac{\sin 30°}{1 + \cos 30°}$$

$$= \frac{\frac{1}{2}}{1 + \frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2\left(1 + \frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1}{2+\sqrt{3}}$$

$$\cot 15° = \frac{1}{\tan 15°}$$

$$= 2 + \sqrt{3}$$

b) $\frac{1}{3\times4\times5} + \frac{2}{4\times5\times6} + \frac{3}{5\times6\times7} + \dots + \frac{n}{(n+2)(n+3)(n+4)} = \frac{1}{6} - \frac{1}{n+3} + \frac{2}{(n+3)(n+4)}$

1) Prove true for $n=1$

LHS $= \frac{1}{3\times4\times5}$ $= \frac{1}{60}$

RHS $= \frac{1}{6} - \frac{1}{4} + \frac{2}{4\times5}$ $= \frac{1}{60}$

$\therefore$ true for $n=1$

2) assume true for $n=k$.

ie.

$$\frac{1}{3\times4\times5} + \frac{2}{4\times5\times6} + \frac{3}{5\times6\times7} + \dots + \frac{k}{(k+2)(k+3)(k+4)} = \frac{1}{6} - \frac{1}{k+3} + \frac{2}{(k+3)(k+4)}$$

3) Prove true for $n=k+1$.

ie.

$$\frac{1}{3\times4\times5} + \frac{2}{4\times5\times6} + \frac{3}{5\times6\times7} + \dots + \frac{k}{(k+2)(k+3)(k+4)} + \frac{k+1}{(k+1+2)(k+1+3)(k+1+4)}$$

$$= \frac{1}{6} - \frac{1}{k+1+3} + \frac{2}{(k+1+3)(k+1+4)}$$

$$= \frac{1}{6} - \frac{1}{k+4} + \frac{2}{(k+4)(k+5)}$$

$$= \frac{1}{6} - \frac{k+5-2}{(k+4)(k+5)}$$

$$= \frac{1}{6} - \frac{k+3}{(k+4)(k+5)}$$

③ correct solution

② proves that $p(k)$ true $\rightarrow p(k+1)$ is true

① verifies initial case

LHS $= \frac{1}{6} - \frac{1}{k+3} + \frac{2}{(k+3)(k+4)} + \frac{k+1}{(k+3)(k+4)(k+5)}$

$$= \frac{1}{6} - \frac{(k+4)(k+5) - 2(k+5) - (k+1)}{(k+3)(k+4)(k+5)}$$

$$= \frac{1}{6} - \frac{k^2+5k+4k+20-2k-10-k-1}{(k+3)(k+4)(k+5)}$$

$$= \frac{1}{6} - \frac{k^2+6k+9}{(k+3)(k+4)(k+5)}$$

$$= \frac{1}{6} - \frac{(k+3)^2}{(k+3)(k+4)(k+5)}$$

$$= \frac{1}{6} - \frac{k+3}{(k+4)(k+5)}$$

= RHS

$\therefore$ true for $n=k+1$. Since true for $n=1$, by PMI, true for $n\geq1$.

c) $\frac{dP}{dt} = \frac{1}{15}P\left(\frac{C-P}{C}\right)$

$\frac{dt}{dP} = \frac{15}{P}\left(\frac{C}{C-P}\right)$

$= 15\times\frac{C}{P(C-P)}$

$= 15\left(\frac{1}{P} + \frac{1}{C-P}\right)$

$t = 15\int \frac{1}{P} + \frac{1}{C-P}\,dP$

$= 15\left[\ln|P| - \ln|C-P|\right] + c$

$= 15\ln\left|\frac{P}{C-P}\right| + c$

④ correct solution

③ uses given information to obtain two equations in A and C

② integrates both sides correctly to obtain $t = \ln\left|\frac{P}{C-P}\right| + c$

① Separates the variables in the differential equation when $t=0$, $P=85\,000$

$$0 = 15\ln\left(\frac{85000}{c-85000}\right)+c$$

$$c = -15\ln\left(\frac{85\,000}{c-85\,000}\right)$$

$$= 15\ln\left(\frac{c-85\,000}{85\,000}\right)$$

when $t=30$, $P=425\,000$

$$30 = 15\ln\left(\frac{425\,000}{c-425\,000}\right) + 15\ln\left(\frac{c-85\,000}{85000}\right)$$

$$2 = \ln\left(\frac{425\,000}{c-425\,000} \times \frac{c-85\,000}{85\,000}\right)$$

$$2 = \ln\left(\frac{425000\,c - (425000 \times 85\,000)}{85\,000\,c - (425\,000 \times 85\,000)}\right)$$

$$2 = \ln\left(\frac{5c - 425\,000}{c - 425\,000}\right)$$

$$e^2 = \frac{5c - 425000}{c - 425\,000}$$

$$e^2(c-425\,000) = 5c - 425\,000$$

$$e^2c - 425000e^2 = 5c - 425000$$

$$e^2c - 5c = 425000e^2 - 425\,000$$

$$c(e^2-5) = 425\,000(e^2-1)$$

$$c = \frac{425\,000(e^2-1)}{e^2-5}$$

$$c = 1\,136\,578.100$$

$\therefore$ carrying capacity $\approx 1\,137\,000$

OR

$$\frac{dP}{dt} = \frac{1}{15}P\left(\frac{c-P}{P}\right)$$

$$\int \frac{c}{P(c-P)}\,dP = \int \frac{1}{15}\,dt$$

$$\int\left(\frac{1}{P}+\frac{1}{c-P}\right)dP = \frac{1}{15}t + c$$

$$[\ln|P| - \ln|c-P|] = \frac{t}{15} + C$$

$$\ln\left|\frac{P}{c-P}\right| = \frac{t}{15} + C$$

$$\frac{P}{c-P} = Ae^{\frac{t}{15}}$$

$t=0$, $P = 85\,000$

$$\frac{85000}{c-85000} = A$$

$t=30$, $P=425\,000$

$$\frac{425\,000}{c-425\,000} = \frac{85\,000}{c-85\,000} \times e^2$$

$$\frac{85}{c-425000} = \frac{17e^2}{c-85\,000}$$

$$85(c-85\,000) = 17e^2(c-425\,000)$$

$$85c - 7225\,000 = 17e^2c - 7225000e^2$$

$$c(85-17e^2) = 7225\,000 - 7225\,000e^2$$

$$c = \frac{7225\,000(1-e^2)}{85-17e^2}$$

$$c = 1136578.1$$

$$c \doteqdot 1137\,000$$

d)

$y = 10\log_e(x-2)$

$\frac{y}{10} = \log_e(x-2)$

$e^{\frac{y}{10}} = x-2$

$x = e^{\frac{y}{10}} + 2$

$x^2 = \left(e^{\frac{y}{10}} + 2\right)^2$

$= \left(e^{\frac{y}{10}}\right)^2 + 4e^{\frac{y}{10}} + 4$

$= e^{\frac{y}{5}} + 4e^{\frac{y}{10}} + 4$

$V = \pi\int_0^{10} x^2 \, dy$

$= \pi\int_0^{10}\left(e^{\frac{y}{5}} + 4e^{\frac{y}{10}} + 4\right) dy$

$= \pi\left[5e^{\frac{y}{5}} + 40e^{\frac{y}{10}} + 4y\right]_0^{10}$

$= \pi\left[\left(5e^{\frac{10}{5}} + 40e^{\frac{10}{10}} + 4(10)\right) - \left(5e^{\frac{0}{5}} + 40e^{\frac{0}{10}} + 4(0)\right)\right]$

$= \pi\left[(5e^2 + 40e + 40) - (5 + 40)\right]$

$= \pi\left(5e^2 + 40e - 5\right)$

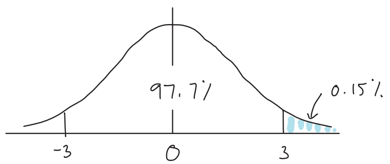
$= 5\pi\left(e^2 + 8e - 1\right)$

③ correct solution

② correct primitive function

① obtains $\pi\int_0^{10}\left(e^{y/5} + 4e^{y/10} + 4\right) dy$

e) A tail with area 0.15% means $\hat{p} = 0.008$ is two standard deviations above the mean so $0.008 = \mu + 3\sigma$



$E(\hat{p}) = p = 0.007$

$P(\hat{p} \geq 0.008) < 0.15\%$

$P(\hat{p} \geq 0.008) = P(z\text{-score} \leq 3)$

③ correct solution

② Recognises 3σ is important and has σ in terms of n

OR Recognise 3σ is important and has the value of σ

$3 \times \sigma = 0.008 - E(\hat{p})$

$= 0.008 - 0.007$

$= 0.001$

$\sigma = \frac{1}{3000}$

$\sigma^2 = \frac{p(1-p)}{n}$

$\left(\frac{1}{3000}\right)^2 = \frac{(0.007)(1-0.007)}{n}$

$n = \frac{(0.007)(1-0.007)}{\left(\frac{1}{3000}\right)^2}$

$n = 62\,559$

$n = 63\,000$

OR

$0.008 = 0.007 + 3\sqrt{\frac{(0.007)(1-0.007)}{n}}$

$0.001 = 3\frac{\sqrt{0.006951}}{\sqrt{n}}$

$\sqrt{n} = 3000\sqrt{0.006951}$

$n = 3000^2 \times 0.006951$

$n = 62559$

$n = 63\,000$

① Finds σ in terms of n

OR sketches a normal distribution and shades correct region

OR writes $P(\hat{p} \geq 0.008) < 0.15\%$